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Journal of Applied Mathematics and Mechanics 71 (2007) 611–621
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

An asymptotic approach to the problem of the free oscillations of a beam☆

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Received 23 September 2006

Abstract

Equations describing the free small longitudinal and transverse oscillations of a straight elastic beam of rectangular cross section are obtained using the plane linear theory of elasticity and the method of integrodifferential relations. The initial system of partial differential equations is reduced to a system of ordinary linear differential equations with constant coefficients. The effect of the geometrical and elastic characteristics of the beam on the frequency and form of the natural oscillations is investigated. For longitudinal motions it is shown that different types of natural displacements and internal stresses of the beam exist. For transverse oscillations, it is found that there are frequency zones corresponding to different forms of the solutions of the characteristic equation obtained using the proposed model.

Among the simplified models proposed for the approximate solutions of boundary-value problems of the mechanics of a deformed solid, the theory of beams, based on intuitive hypotheses, proposed by Bernoulli at the end of the 17th century[1] occupies an important place. Despite the fact that this theory is applicable to a wide range of problems, it does not take into account such important mechanical characteristics of elastic structures as the effect of the shear and anisotropic properties of the material on the stress-strain state. More-accurate formulae, which enable the effect of Poisson's ratio to be taken into account, have been proposed both for static problems (a Timoshenko beam[2]) and for dynamic problems (the Rayleigh correction[3]). Below, to derive the equations of the free oscillations of a beam, we use an approach based on an expansion of the unknown stress and displacement functions in terms of a small parameter (the ratio of the structural height of the beam to its length).[4]

1. Formulation of the boundary-value problem of elasticity

We will consider an elastic rectangular plate with dimensions of $a \times l$ of unit thickness. We will introduce a Cartesian system of coordinates Oxy, the origin of which is situated in the middle of one of the sides of the plate of length a, while the x axis is parallel to the sides of length l. We will confine the analysis of the stress-strain state of an isotropic

☆ *Prikl. Mat. Mekh.* Vol. 71, No. 4, pp. 670–680, 2007.
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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2007.09.014

body to the two-dimensional case, described by the system of differential equations of the linear theory of elasticity:[5]

$$\varepsilon_x^0 = \frac{1}{E}(\sigma_x - \nu\sigma_y), \quad \varepsilon_y^0 = \frac{1}{E}(\sigma_y - \nu\sigma_x), \quad \varepsilon_{xy}^0 = \frac{\tau_{xy}}{2G}, \quad G = \frac{E}{2(1+\nu)} \tag{1.1}$$

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + f_x = 0, \quad \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + f_y = 0 \tag{1.2}$$

$$\varepsilon_x^0 = \frac{\partial u}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy}^0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{1.3}$$

Assuming that the boundary of the plate is load-free, we will write the boundary conditions in the form

$$\sigma_x n_x + \tau_{xy} n_y = 0, \quad \tau_{xy} n_x + \sigma_y n_y = 0 \tag{1.4}$$

Here ε_x^0, ε_y^0 and ε_{xy}^0 are the components of the strain tensor ε^0, σ_x, σ_y and τ_{xy} are the components of the stress tensor $\boldsymbol{\sigma}$, u and v are the components of the displacement vector $\mathbf{u}$, f_x and f_y are the components of the vector of the bulk forces $\mathbf{f}$, n_x and n_y are the components of the unit vector of the normal to the boundary $\mathbf{n}$, E is the modulus of elasticity, G is the shear modulus and ν is Poisson's ratio.

We will assume that the body in question may execute small elastic oscillations about the equilibrium position and the vector $\mathbf{f}$ is determined by inertial forces, due to the motion of points of the elastic body:

$$f_x = -\rho\frac{\partial^2 u}{\partial t^2}, \quad f_y = -\rho\frac{\partial^2 v}{\partial t^2} \tag{1.5}$$

where ρ is the volume density of the body.

When the length of the plate is much greater than its width a, its stress-strain state is described by the approximate equations of the classical theory of beams. The equations of the longitudinal and transverse oscillations of the beam[1,2] have the form

$$E\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} = 0, \quad EI\frac{\partial^4 v}{\partial x^4} + \rho S\frac{\partial^2 v}{\partial t^2} = 0 \tag{1.6}$$

where EI ($I = a^3/12$) is the bending stiffness, while S is the area of transverse cross section of the beam.

2. The method of integrodifferential relations

Using the method of integrodifferential relations[6–8] the local linear relations (1.1) can be replaced by the integral equality

$$\Phi = \int_{-a/2}^{a/2}\int_0^l \left[\left(\varepsilon_x^0 - \frac{\sigma_x - \nu\sigma_y}{E}\right)^2 + \left(\varepsilon_y^0 - \frac{\sigma_y - \nu\sigma_x}{E}\right)^2 + 2\left(\varepsilon_{xy}^2 - \frac{\tau_{xy}}{2G}\right)^2\right] dxdy = 0 \tag{2.1}$$

Unlike the classical formulation (1.1)–(1.3), in this case the unknown functions are the components of the stress tensor $\boldsymbol{\sigma}$ and the displacement vector $\mathbf{u}$. To solve the problem of elasticity, we will introduce the equivalent variational formulation

$$\Phi[\boldsymbol{\sigma}, \mathbf{u}] \to \min_{\boldsymbol{\sigma}, \mathbf{u}} \tag{2.2}$$

when relations (1.2)–(1.4) are strictly satisfied.

In order to determine the natural oscillations of the elastic body in question, we use the method of separation of variables and represent the unknown functions of the components of the stress tensor and the displacement vector in the form of infinite series in powers of $Y = \frac{y}{l}$

$$\sigma_x(x,y) = e^{i\omega t}\sum_{n=0}^{\infty}\sigma_x^{(n)}(x)Y^n, \quad \sigma_y(x,y) = e^{i\omega t}(1-\eta^{-2}Y^2)\sum_{n=0}^{\infty}\sigma_y^{(n)}(x)Y^n$$

$$\tau_{xy}(x,y) = e^{i\omega t}(1-\eta^{-2}Y^2)\sum_{n=0}^{\infty}\tau_{xy}^{(n)}(x)Y^n \tag{2.3}$$

$$u(x,y) = e^{i\omega t}\sum_{n=0}^{\infty}u_n(x)Y^n, \quad \upsilon(x,y) = e^{i\omega t}\sum_{n=0}^{\infty}\upsilon_n(x)Y^n$$

where ω is the unknown frequency of natural oscillations and $\eta = a/(2l)$ is a geometrical parameter of the plate.

This choice of the expansion of the unknown functions in powers of Y is made for the following reasons: the coefficients of corresponding powers of Y have the dimensions of stresses or displacements, and in the case of a narrow plate ($a \ll l$) the expansion is carried out with respect to the small parameter η, since $|y| \leq \frac{a}{2}$. Moreover, the boundary conditions on the upper and lower parts of the plate, which have the form

$$\sigma_y(x, \pm a/2) = \tau_{xy}(x, \pm a/2) = 0 \tag{2.4}$$

are satisfied automatically.

Taking the expansion of the functions of the stresses and displacements (2.3) into account, we will write the function Φ (2.1) in the form

$$\Phi = \int_{-a/2}^{a/2}\int_{0}^{l}[H_1^2 + H_2^2 + 2H_3^2]dxdy \tag{2.5}$$

where

$$H_1 = \sum_{n=0}^{\infty}\left[\frac{du_n}{dx} - \frac{\sigma_x^{(n)}}{E} + \frac{\nu\sigma_y^{(n)}}{E}(1-\eta^{-2}Y^2)\right]Y^n$$

$$H_2 = \sum_{n=0}^{\infty}\left[\frac{n+1}{l}\upsilon_{n+1} + \frac{\nu\sigma_x^{(n)}}{E} - \frac{\sigma_y^{(n)}}{E}(1-\eta^{-2}Y^2)\right]Y^n \tag{2.6}$$

$$H_3 = \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{n+1}{l}u_{n+1} + \frac{d\upsilon_n}{dx} - \frac{\tau_{xy}^{(n)}}{G}(1-\eta^{-2}Y^2)\right]Y^n$$

If integral (2.5) is equal to zero, this means that the functions A_α ($\alpha = 1, 2, 3$) are equal to zero in the region occupied by the plate, everywhere with the exception, possibly, of a set of points of measure zero.[6] In this paper we will not assume that such singular points exist, and we will therefore henceforth assume that, if $\Phi = 0$, then $H_\alpha \equiv 0$ ($\alpha = 1, 2, 3$) in the region occupied by the plate. Hence it follows that all the coefficients of powers of Y in relations (2.6) are equal to zero.

After substituting representations (2.3) for the stresses and displacements into the equilibrium Eq. (1.2), they can be solved for the displacement functions $u_n(x)$ and $\upsilon_n(x)$

$$u_n(x) = -\frac{1}{\rho\omega^2}\left(\frac{d\sigma_x^{(n)}}{dx} + \frac{n+1}{l}\left[\tau_{xy}^{(n+1)} - \frac{1-\delta_{0n}}{\eta^2}\tau_{xy}^{(n-1)}\right]\right)$$

$$\upsilon_n(x) = -\frac{1}{\rho\omega^2}\left(\frac{d\tau_{xy}^{(n)}}{dx} - \frac{(1-\delta_{0n})(1-\delta_{1n})}{\eta^2}\frac{d\tau_{xy}^{(n-2)}}{dx} + \frac{n+1}{l}\left[\sigma_y^{(n+1)} - \frac{1-\delta_{0n}}{\eta^2}\sigma_y^{(n-1)}\right]\right) \tag{2.7}$$

taking expressions (1.5) into account, where δ_{ij} is the Kronecker delta and $n = 0, 1, 2, \ldots$.

Substituting the displacement functions (2.7) into relations (2.6) and grouping in powers of Y, we obtain the following representations of the functions H_α:

$$H_\alpha = \sum_{n=0}^{\infty} h_{\alpha n} Y^n, \quad \alpha = 1, 2, 3 \tag{2.8}$$

where

$$\begin{aligned}
h_{1n} &= -\frac{1}{\rho\omega^2}\left(\frac{d^2\sigma_x^{(n)}}{dx^2} + \frac{n+1}{l}\left[\frac{d\tau_{xy}^{(n+1)}}{dx} - \frac{1-\delta_{0n}}{\eta^2}\frac{d\tau_{xy}^{(n-1)}}{dx}\right]\right) - \\
&- \frac{\sigma_x^{(n)}}{E} + \frac{\nu\sigma_y^{(n)}}{E} - \frac{(1-\delta_{0n})(1-\delta_{1n})}{\eta^2}\frac{\nu\sigma_y^{(n-2)}}{E} \\
h_{2n} &= -\frac{n+1}{\rho\omega^2 l}\left(\frac{d\tau_{xy}^{(n+1)}}{dx} - \frac{1-\delta_{0n}}{\eta^2}\frac{d\tau_{xy}^{(n-1)}}{dx} + \frac{n+2}{l}\left[\sigma_y^{(n+2)} - \frac{\sigma_y^{(n)}}{\eta^2}\right]\right) + \\
&+ \frac{\nu\sigma_x^{(n)}}{E} - \frac{\sigma_y^{(n)}}{E} + \frac{(1-\delta_{0n})(1-\delta_{1n})}{\eta^2}\frac{\sigma_y^{(n-2)}}{E} \\
2h_{3n} &= -\frac{n+1}{\rho\omega^2 l}\left(\frac{d\sigma_x^{(n+1)}}{dx} + \frac{n+2}{l}\left[\tau_{xy}^{(n+2)} - \frac{\tau_{xy}^{(n)}}{\eta^2}\right]\right) - \\
&- \frac{1}{\rho\omega^2}\left(\frac{d^2\tau_{xy}^{(n)}}{dx^2} + \frac{n+1}{l}\left[\frac{d\sigma_y^{(n+1)}}{dx} - \frac{1-\delta_{0n}}{\eta^2}\frac{d\sigma_y^{(n-1)}}{dx}\right]\right) - \\
&- \frac{\tau_{xy}^{(n)}}{G} + \frac{(1-\delta_{0n})(1-\delta_{1n})}{\eta^2}\left(\frac{\tau_{xy}^{(n-2)}}{G} + \frac{1}{\rho\omega^2}\frac{d^2\tau_{xy}^{(n-2)}}{dx^2}\right), \quad n = 0, 1, 2, \ldots
\end{aligned} \tag{2.9}$$

Analysing the structure of the coefficients $h_{\alpha n}$ in expressions (2.9), we can distinguish two independent subsystems of unknown stress functions, one of which corresponds to tension and compression of the plate, while the other corresponds to bending of the plate. It was pointed out in Ref. 4 that such a decomposition is characteristic for any power of the approximation of the stress and displacement functions and arises due to the symmetry of the plate about the x axis. In this case the functions σ_x^{2n}, σ_y^{2n}, τ_{xy}^{2n+1}, u_{2n}, v_{2n+1} describe tension and compression, while the functions σ_x^{2n+1}, σ_y^{2n+1}, τ_{xy}^{2n}, u_{2n+1}, v_{2n} describe bending of the plate; the coefficients $h_{1,(2n)}$, $h_{2,(2n)}$, $h_{3,(2n+1)}$ correspond to longitudinal oscillations while the coefficients $h_{1,(2n+1)}$, $h_{2,(2n+1)}$, $h_{3,(2n)} (n = 0, 1, 2, \ldots)$ correspond to transverse oscillations.

In order to obtain an approximate solution of problem (1.2)–(1.4), (2.1), we will consider a finite-dimensional representation of the stress and displacement functions in (2.3) ($n \le N$, where N is the maximum power of the expansion in Y). This indicates that the remaining functions ($n > N$) in (2.3) can be assumed to be equal to zero. Then the number of non-zero coefficients $h_{\alpha m}$ will also be finite and equal to $3(N+3)$ ($\alpha = 1, 2, 3$, $m = 0, 1, \ldots, N+2$), and hence it is not possible to solve the system of $3(N+3)$ ordinary differential equations $h_{\alpha m} = 0$ for the $3(N+1)$ unknown stress functions σ_x^n, σ_y^n, τ_{xy}^n. Using the variational formulation (2.2), we can employ different methods of optimization[4] to solve this over-determined system. In this paper we propose an approach in which the equations corresponding to lower powers of the expansion in Y of the functions $H_\alpha (\alpha = 1, 2, 3)$ are accurately satisfied. The remaining non-zero coefficients $h_{\alpha m}$ are used to estimate the quality of the approximate solution of the integrodifferential problem (1.2)–(1.4), (2.1).

3. Longitudinal oscillations of the beam

Using decomposition of the unknown stress functions and applying the approach proposed in the previous section, we obtain a finite-dimensional system of ordinary differential equations

$$\begin{aligned} &h_{10} = 0, \quad h_{20} = 0 \\ &h_{1,(2n)} = 0, \quad h_{2,(2n)} = 0, \quad h_{3,(2n-1)} = 0, \quad n = 1, 2, \dots, N_1 + 1 \end{aligned} \tag{3.1}$$

and boundary conditions

$$\begin{aligned} &\sigma_x^{(0)}(0) = \sigma_x^{(0)}(l) = 0 \\ &\sigma_x^{(2n)}(0) = \sigma_x^{(2n)}(l) = \tau_{yx}^{(2n-1)}(0) = \tau_{yx}^{(2n-1)}(l) = 0, \quad n = 1, 2, \dots, N_1 \end{aligned} \tag{3.2}$$

which describe the longitudinal oscillations of the beam. Note that all the stress functions $\sigma_y^{(2n)}$ occur algebraically in the coefficients $h_{2,(2n)}$ and can be expressed in explicit form from system (3.1) in terms of $\sigma_x^{(2n)}$, $\tau_{xy}^{(2n-1)}$ and their derivatives. The first $2N_1 + 1$ second-order differential equations

$$h_{10} = 0, \quad h_{1,(2n)} = 0, \quad h_{3,(2n-1)} = 0, \quad n = 1, 2, \dots, N_1 \tag{3.3}$$

after substituting $\sigma_y^{(2n)}$ into them are solved for the $2N_1 + 1$ unknowns $\sigma_x^{(2n)}$, $\tau_{xy}^{(2n-1)}$. The general solution of system (3.3) obtained, taking boundary conditions (3.2) into account, are used to find the eigenvalues ω from the condition for non-trivial solutions to exist. Henceforth we will call the system of equations (3.3) with boundary conditions (3.2) the N_1-th approximation of the original problem.

Substituting the solution $\tilde{\sigma}_y^{(2n)}(x)$, $\tilde{\sigma}_x^{(2n)}(x)$, $\tilde{\tau}_{xy}^{(2n-1)}(x)$ of system (3.3) into (2.5) and bearing in mind the finite-dimensional representation of the stress and displacement fields, we obtain

$$\begin{aligned} &\tilde{\Phi}_{N_1} = \int_{-a/2}^{a/2} \int_0^l [\tilde{H}_{1,(N_1)}^2 + \tilde{H}_{2,(N_1)}^2 + 2\tilde{H}_{3,(N_1)}^2] dx dy \ge 0 \\ &\tilde{H}_{1,(N_1)} = -\frac{\nu\tilde{\sigma}_y^{(2N_1)}}{E\eta^2} Y^{2N_1+2}, \quad \tilde{H}_{2,(N_1)} = \frac{\tilde{\sigma}_y^{(2N_1)}}{E\eta^2} Y^{2N_1+2} \\ &\tilde{H}_{3,(N_1)} = \left[\frac{1}{\rho\omega^2}\frac{d^2\tilde{\tau}_{xy}^{(2N_1-1)}}{dx^2} + \frac{2(N_1+1)}{\rho\omega^2 l}\frac{d\tilde{\sigma}_y^{(2N_1)}}{dx} + \frac{\tilde{\tau}_{xy}^{(2N_1-1)}}{G}\right]\frac{Y^{2N_1+1}}{\eta^2} \end{aligned} \tag{3.4}$$

The value of the integral $\tilde{\Phi}_{N_1}$ can serve as a criterion of the quality of the approximate solution.

As an example, we will consider the zero approximation of the problem of the longitudinal oscillations of a rectangular plate ($N_1 = 0$). In this case, we only need to take into account the following two equations in system (3.1)

$$\frac{E}{\rho\omega^2}\frac{d^2\sigma_x^{(0)}}{dx^2} + \sigma_x^{(0)} - \nu\sigma_y^{(0)} = 0, \quad \frac{8\sigma_y^{(0)}}{\rho\omega^2 a^2} - \frac{\sigma_y^{(0)} - \nu\sigma_x^{(0)}}{E} = 0 \tag{3.5}$$

System (3.5) reduces to a single second-order ordinary differential equation in the unknown function $\sigma_x^{(0)}$

$$\frac{d^2\sigma_x^0}{dx^2} + \lambda^2\sigma_x^{(0)} = 0 \tag{3.6}$$

where

$$\lambda^2 = \frac{\omega^2}{\omega_0^2 a^2}\left(1 - \frac{\omega^2\nu^2}{\omega^2 - 8\omega_0^2}\right) \tag{3.7}$$

Here ω_0 is characteristic frequency, defined by the following relation: $\omega_0^2 = E/(\rho a^2)$. The quantity $\sigma_y^{(0)}$ is then found from the second equation in (3.5)

$$\sigma_y^{(0)} = \frac{\nu\omega^2}{\omega^2 - 8\omega_0^2}\sigma_x^{(0)} \tag{3.8}$$

The boundary conditions are written in the form

$$\sigma_x^{(0)}(0) = \sigma_x^{(0)}(l) = 0 \tag{3.9}$$

Unlike the classical equation of the longitudinal oscillations of a beam (the first equation of (1.6)) in terms of the displacement function u, relation (3.6) contains the parameter λ, which depends non-linearly on the frequency of natural oscillations ω, and also on the parameters of the problem ν, a and ω_0^2. Note that the quantities $\lambda^2(\omega)$ are positive when $\omega \in (0, \omega_1) \cup (\omega_2, \infty)$ and negative when $\omega \in (\omega_1, \omega_2)$, where

$$\omega_1^2 = 8\omega_0^2, \quad \omega_2^2 = 8\omega_0^2/(1-\nu^2) \tag{3.10}$$

It can be shown that when $\lambda^2(\omega) \leq 0$ there are non-trivial solutions of the eigenvalue problem (3.6), (3.9). When $\lambda^2(\omega) > 0$ the solution has the form

$$\sigma_x^{(0)} = c\sin(\lambda x) \tag{3.11}$$

The characteristic equation for determining the natural frequencies ω can be written as follows:

$$\frac{\omega}{\omega_0}\sqrt{1 - \frac{\omega^2\nu^2}{\omega^2 - 8\omega_0^2}} = \xi n, \quad \xi = \frac{\pi a}{l}, \quad n \in \mathbb{N} \tag{3.12}$$

The two positive roots ω_+ and ω_- of Eq. (3.12) are found in explicit form as functions which depend on $n \geq 0$

$$\omega_\pm(n) = \sqrt{\frac{8 + \xi^2 n^2 \pm \sqrt{64 + \xi^2 n^2[\xi^2 n^2 - 16(1 - 2\nu^2)]}}{2(1-\nu^2)}}\omega_0 \tag{3.13}$$

Here the functions $\omega_\pm(n)$ are monotonically increasing, $\omega_-(0) = 0$, $\omega_+(0) = \omega_2$ and the following asymptotic relations hold for large values of n

$$\omega_-|_{n\to\infty} \to \omega_1, \quad \omega_+|_{n\to\infty} \to \omega_0\xi n(1-\nu^2)^{-1/2}$$

In Fig. 1 we show the functions $\omega_\pm(n)$ (the continuous curves) for the following values of the system parameters: $\omega_0 = 1$, $\xi = \pi/10$, $\nu = 0.3$. The horizontal dashed lines represent the values $\omega = \omega_1 = 2\sqrt{2} \approx 2.828$ and $\omega = \omega_2 \approx 2.965$. The inclined dash-dot line corresponds to the classical beam solution for the same system parameters. The natural frequencies are found from the above relations for integer n. Their values are presented below for the first ten n:

n	1	2	3	4	5	6	7	8	9	10
ω_-	0.314	0.627	0.937	1.243	1.542	1.827	2.089	2.312	2.480	2.591
ω_+	2.967	2.972	2.981	2.997	3.021	3.059	3.121	3.223	3.380	3.595

Note that the form of the behaviour of the function $\omega_-(n)$ for the first few values of n were validated for the first time by Rayleigh.[3]

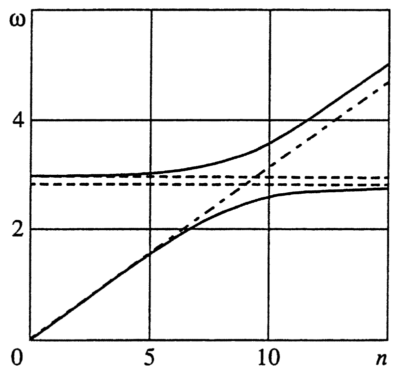

Fig. 1.

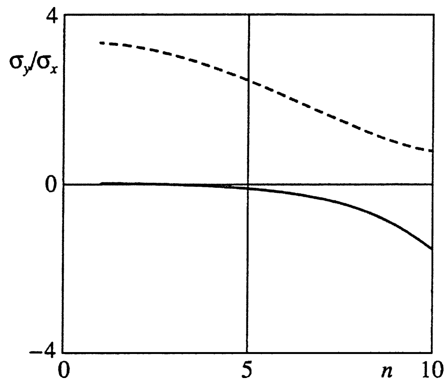

Fig. 2.

The dependence of the ratios of the amplitudes of the eigenfunctions $\sigma_y^{(0)}/\sigma_x^{(0)}$ on the number n of the natural frequency is shown in Fig. 2. The continuous curve corresponds to the lower branch of the solution ω_-, while the dashed curve corresponds to the upper branch ω_+. The essential feature of the longitudinal oscillations is the fact that, for the lower branch $\omega_-(n)\,\sigma_y^{(0)}/\sigma_x^{(0)} \le 0$, while for the upper branch $\sigma_y^{(0)}/\sigma_x^{(0)} \ge 0$. Note also that for given parameters of the system and $n = 9$ the maximum values of $\sigma_x^{(0)}$ and $\sigma_y^{(0)}$ are almost equal for both branches of the solution (3.13).

4. Transverse oscillations of a beam

In the same way as for longitudinal motions, using decomposition, we can write the following system of ordinary differential equations

$$h_{1,(2n+1)} = 0, \quad h_{2,(2n+1)} = 0, \quad h_{3,(2n)} = 0, \quad n = 0, 1, \ldots, N_2 + 1 \tag{4.1}$$

and boundary conditions

$$\sigma_x^{(2n+1)}(0) = \sigma_x^{(2n+1)}(l) = \tau_{yx}^{(2n)}(0) = \tau_{yn}^{(2n)}(l) = 0, \quad n = 0, 1, \ldots, N_2 \tag{4.2}$$

which describe transverse oscillations of a plate. The stress functions $\sigma_y^{(2n+1)}$ are expressed in explicit form from system (4.1) in terms of $\sigma_x^{(2n+1)}$, $\tau_{xy}^{(2n)}$ and their derivatives. The first $2N_2 + 2$ second-order differential equations

$$h_{1,(2n+1)} = 0, \quad h_{3,(2n)} = 0, \quad n = 0, 1, \ldots, N_2 \tag{4.3}$$

after substituting $\sigma_y^{(2n+1)}$ into them are solved for the $2N_2+2$ unknowns $\sigma_x^{(2n+1)}$, $\tau_{xy}^{(2n)}$. As in the case of longitudinal oscillations, the general solution of system (4.3), taking boundary conditions (4.2) into account, is used to find the eigenvalues ω.

Substituting the solution $\tilde{\sigma}_x^{(2n+1)}(x)$, $\tilde{\tau}_{xy}^{(2n)}(x)$ of system (4.3) and the corresponding functions $\tilde{\sigma}_y^{(2n+1)}(x)$ into functional (2.5), we obtain

$$\tilde{\Phi}_{N_2} = \int_{-a/2}^{a/2}\int_0^l [\tilde{H}_{1,(N_2)}^2 + \tilde{H}_{2,(N_2)}^2 + 2\tilde{H}_{3,(N_2)}^2]dxdy \geq 0$$

$$\tilde{H}_{1,(N_2)} = -\frac{\nu\tilde{\sigma}_y^{(2N_2+1)}}{E\eta^2}Y^{2N_2+3}, \quad \tilde{H}_{2,(N_2)} = \frac{\tilde{\sigma}_y^{(2N_2+1)}}{E\eta^2}Y^{2N_2+3} \tag{4.4}$$

$$\tilde{H}_{3,(N_2)} = \left[\frac{1}{\rho\omega^2}\frac{d^2\tilde{\tau}_{xy}^{(2N_2)}}{dx^2} + \frac{2(N_2+3)}{\rho\omega^2 l}\frac{d\tilde{\sigma}_y^{(2N_2+1)}}{dx} + \frac{\tilde{\tau}_{xy}^{(2N_2)}}{G}\right]\frac{Y^{2N_2+2}}{\eta^2}$$

The value of the integral $\tilde{\Phi}_{N_2}$ serves as a criterion of the quality of the approximate solution of the problem of the transverse oscillations of a plate.

When $N_2 = 0$ (the zero approximation) the system of Eq. (4.3) has the form

$$\frac{d^2\sigma_x^{(1)}}{dx^2} - \frac{8l}{a^2}\frac{d\tau_{xy}^{(0)}}{dx} + \frac{\rho\omega^2}{E}(\sigma_x^{(1)} - \nu\sigma_y^{(1)}) = 0$$

$$\frac{d^2\tau_{xy}^{(0)}}{dx^2} + \frac{1}{l}\left(\frac{d\sigma_x^{(1)}}{dx} + \frac{d\sigma_y^{(1)}}{dx}\right) + \left(\frac{\rho\omega^2}{G} - \frac{8}{a^2}\right)\tau_{xy}^{(0)} = 0 \tag{4.5}$$

where

$$\sigma_y^{(1)} = \frac{1}{\rho\omega^2a^2 - 24E}\left[\rho\omega^2a^2\nu\sigma_x^{(1)} + 8El\frac{d\tau_{xy}^{(0)}}{dx}\right]$$

Expressing, for example, the function $\tau_{xy}^{(0)}$ from system (4.5) in explicit form

$$\tau_{xy}^{(0)} = A\left[\frac{a^2}{24}\left(2(1+\nu) - \frac{\omega^2}{2\omega_3^2}\right)\frac{d^3\sigma_x^{(1)}}{dx^3} + \left(1 + \nu + \frac{1-2\nu}{6}\frac{\omega^2}{\omega_3^2} - \frac{1-\nu}{12}\frac{\omega^4}{\omega_3^4}\right)\frac{d\sigma_x^{(1)}}{dx}\right]$$

$$A = \frac{3a^2\omega_3^4}{4(1+\nu)l(\omega_3^2 - \omega^2)(6\omega_3^2 - \omega^2)}, \quad \omega_3^2 = \frac{4\omega_0^2}{1+\nu} \tag{4.6}$$

we obtain the following fourth-order differential equation for finding $\sigma_x^{(1)}$

$$\left(1 - \frac{\omega^2}{\omega_4^2}\right)\frac{d^4\sigma_x^{(1)}}{dx^4} + \left[\frac{4(1-\nu)\omega^2}{(1+\nu)a^2\omega_3^2}\left(1 - \frac{\omega^2}{4\omega_3^2}\right) + \frac{12\omega^2}{a^2\omega_3^2}\left(1 - \frac{\omega^2}{24\omega_0^2}\right)\right]\frac{d^2\sigma_x^{(1)}}{dx^2} +$$

$$+ \frac{\rho\omega^2}{EI}\left(1 - \frac{\omega^2}{\omega_3^2}\right)\left(1 - \frac{\omega^2}{\omega_5^2}\right)\sigma_x^{(1)} = 0; \quad \omega_4^2 = 16\omega_0^2, \quad \omega_5^2 = \frac{24\omega_0^2}{1-\nu^2} \tag{4.7}$$

Note that when $\omega = \omega_3$ and $\omega = \sqrt{6}\omega_3$ the denominator in the expression for A vanishes. These cases must be considered separately.

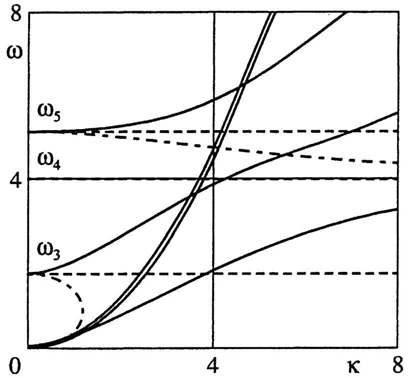

Fig. 3.

Taking expression (4.6) into account, we will write the boundary conditions for Eq. (4.7) in the form

$$\sigma_x^{(1)}(0) = \sigma_x^{(1)}(l) = \tau_{xy}^{(0)}(0) = \tau_{xy}^{(0)}(l) = 0 \tag{4.8}$$

The roots $\kappa_i(\omega)$ ($i = 1, 2, 3, 4$) of the biquadratic characteristic equation for (4.7) are found analytically:

$$\kappa^2 = -\frac{\omega}{2(\omega^2 - 16\omega_0^2)\omega_0^2 a^2}[(1+\nu)(\nu-3)\omega^3 + 32(2+\nu)\omega_0^2\omega \pm$$
$$\pm((1+\nu)^4\omega^6 - 32(1+2\nu)(1+\nu)\omega_0^2\omega^4 + 256(6\nu^2+4\nu-1)\omega_0^4\omega^2 + 3\cdot 8^4\omega_0^6)^{1/2}] \tag{4.9}$$

It can be seen that for all $\omega > 0$, two pure imaginary complex-conjugate roots exist.

Consider the following frequency bands

$$1)\ \omega \in (0, \omega_3), \quad 2)\ \omega \in (\omega_3, \omega_4), \quad 3)\ \omega \in (\omega_4, \omega_5), \quad 4)\ \omega \in (\omega_5, +\infty)$$

In the first and third bands the two remaining roots take real values, while in the second and fourth bands they take pure imaginary values. Consequently, the general solution of Eq. (4.7) for the frequencies of the first and third bands has the form

$$\sigma_x^{(1)} = c_1\sin(|\kappa_1|x) + c_2\cos(|\kappa_1|x) + c_3\operatorname{sh}(|\kappa_2|x) + c_4\operatorname{ch}(|\kappa_2|x) \tag{4.10}$$

while for the second and fourth bands

$$\sigma_x^{(1)} = c_1\sin(|\kappa_1|x) + c_2\cos(|\kappa_1|x) + c_3\sin(|\kappa_2|x) + c_4\cos(|\kappa_2|x) \tag{4.11}$$

The eigenvalues ω are found from the condition for non-trivial solutions (4.10) and (4.11) to exist when boundary conditions (4.8) are satisfied.

The relation between the roots κ_i of the characteristic equation and the frequency ω is shown in Fig. 3 for the dimensionless values $\omega_0 = 1$, $a = 1$ and $\nu = 0.3$. The dependence of the frequency ω on the absolute values of the pure imaginary roots κ_i is represented by the continuous curves, while the dependence on the real values is represented by the dash-dot curves. The horizontal dashed straight lines correspond to the critical values $\omega = \omega_n$ ($n = 3, 4, 5$). The double curve corresponds to the classical solution of the problem of natural flexural oscillations (the second equation of (1.6)) for the same values of the parameters.

Below we give the numerical values of the first 20 natural frequencies of free transverse oscillations of a rectangular plate if length $l = 10$:

n	1	2	3	4	5	6	7	8	9	10
ω	0.062	0.161	0.291	0.441	0.602	0.770	0.941	1.113	1.283	1.450
n	11	12	13	14	15	16	17	18	19	20
ω	1.610	1.741	1.807	1.836	1.962	1.970	2.132	2.142	2.307	2.334

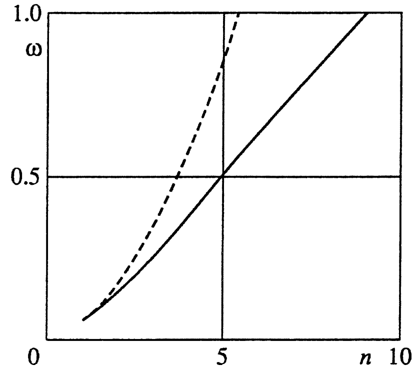

Fig. 4.

The first 12 frequencies correspond to solution (4.10) (the first band), while the remaining frequencies correspond to solution (4.11) (the second band). For the chosen parameters $\omega_3 \approx 1.75$. It can be seen that the number of natural frequencies, corresponding to solution (4.10), is finite and increases as the parameter η decreases.

In Fig. 4 the continuous curve represents the dependence of the first natural frequencies on the frequency number n. The dashed curve corresponds to the frequencies of transverse oscillations, obtained using the classical model of the beam, described by the second equation of (1.6) for the same system parameters. Note the quadratic increase in the classical eigenvalues as n increases, whereas the frequencies obtained using the method of integrodifferential relations depend on n almost linearly and differ considerably from the classical values even for small n numbers.

The natural forms of displacements $u(x, a/2)$ and $v(x, 0)$, obtained using relations (2.7), for different values of $\omega(n)$ with the chosen parameters of the problem, are shown in Fig. 5 ($\omega < \omega_3$). The functions of the bending of the centre line of the plate $v(x, 0)$, corresponding to the natural stresses $\sigma_x^{(1)}$, determined in solution (4.10) with $c_1 = 1$ for $n = 1$ and $n = 12$, are represented by the continuous curves in Fig. 5. The curve with a single maximum corresponds to the case $n = 1$, while the curves with multiple extrema correspond to $n = 12$. The dashed curves represent the horizontal displacements of the upper boundary of the beam $u(x, a/2)$ for different n. For $n = 1$ the form of the natural oscillations is determined by pure bending (there are practically no shear deformations), whereas as n increases shear begins to have a considerable effect on the form of the natural oscillations. Thus, for $n = 12$ the function $u(x, a/2)$ is positive everywhere.

When the parameter η is reduced the difference between the first natural frequencies, obtained by the method of integrodifferential relations (ω) and by the classical approach (ω_c) decreases. Below we present numerical values of the first 6 frequencies ω and ω_c for the following parameters of the problem: $\omega_0 = 1$, $a = 1$, $l = 100$ and $\nu = 0.3$:

n	1	2	3	4	5	6
$\omega \times 10^3$	0.645	1.778	3.482	5.747	8.569	11.94
$\omega_c \times 10^3$	0.646	1.780	3.490	5.769	8.619	12.04

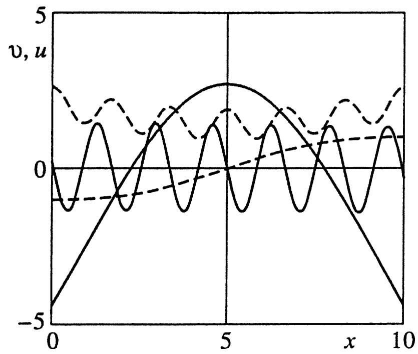

Fig. 5.

Acknowledgement

This research was supported financially by the Russian Foundation for Basic Research (05-01-00563, 05-08-18094) and by the State Support of the Leading Scientific Schools of the Russian Federation (NSh-1245.2006.1, NSh-9831.2006.1).

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Translated by R.C.G.